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ISOMETRIES AND ALMOST ISOMETRIES BETWEEN SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT
We characterize the isometries from $C(X)$ into $C(Y)$ where X and Y are compact metric spaces. We give necessary and sufficient conditions on an isometry from a subset of $C(X)$ into $C(Y)$ to have an extension to the whole space. It is also shown that an almost isometry from the unit ball of $C(X)$ into the unit ball of $C(Y)$ is near to an isometry.

Introduction

A well-known theorem in the theory of Banach spaces states that an isometry of one normed (real-linear) space *onto* another which carries 0 to 0 is linear. The proof is due to Mazur and Ulam and can be found in Banach's book [2].

In this paper we restrict ourselves to the Banach space $C(K)$ where K will denote a compact metric space.

In Section 1 we describe the isometries $F: C(X) \xrightarrow{\text{into}} C(Y)$ and prove that there is a subset $K \subset Y$ such that the map $f \to F(f)|_K$ is a linear isometry. This section also contains an extension theorem for isometries from subsets of $C(X)$ into $C(Y)$.

In Section 2 we deal with the following question: When is an almost isometry from the unit ball of one Banach space into the unit ball of another near to an exact isometry? It will be proved that for $C(K)$-spaces this question has a positive answer.

For linear maps a positive answer has been obtained for $C(K)$-spaces by Benyamini [3] and for L_p-spaces by Alspach [1].

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1. Isometries from $C(X)$ into $C(Y)$

Let X and Y be compact metric spaces with metrics d_1 and d_2. We let $B(z,r)$ denote the ball with center z and radius r.

THEOREM 1.1. *Let F: $C(X)\to C(Y)$ be an isometry with $F(0)=0$. Then there exists a subset $K\subset Y$ such that the map $f\to F(f)|_K$ is a linear isometry where $F(f)|_K$ denotes the restriction of $F(f)$ to K.*

THEOREM 1.2. *Let F: $C(X)\to C(Y)$ be an isometry with $F(0)=0$. Then there exists a linear projection P: $C(Y)\to C(Y)$ such that $P\circ F$ is a linear isometry of $C(X)$ into $C(Y)$.*

For $\forall x\in X$ we define $f_x\in C(X)$ by:

$$f_x(r)=1-\frac{d_1(x,r)}{\operatorname{diam}(X)},\qquad r\in X$$

and we define A_x to be the set of $y\in Y$ for which

$$|F(f_x)(y)|=1$$

and

$$F(tf_x)(y)=tF(f_x)(y)\qquad \text{for } \forall t\in R.$$

In order to prove Theorems 1.1 and 1.2 and for use later on we prove the following lemmas;

LEMMA 1.1. *The set A_x is nonempty for $\forall x\in X$.*

LEMMA 1.2. *For every $g\in C(X)$ and every $y\in A_x$ we have*

$$F(g)(y)=g(x)F(f_x)(y).$$

LEMMA 1.3. *$\bigcup_{x\in X}A_x$ is closed.*

PROOF OF LEMMA 1.1. Let $x\in X$. For each $0<s<\infty$ let

$$Y_s=\{y\in Y: |F(sf_x)(y)-F(-sf_x)(y)|=2s\}.$$

If $t<s$ then $Y_s\subset Y_t$.

To prove this let $y\in Y_s$ and assume without loss of generality that $F(sf_x)(y)=s$. Then $s-t=\|F(sf_x)-F(tf_x)\|\geqq|s-F(tf_x)(y)|$ which implies that $F(tf_x)(y)\geqq t$. Similarly $F(-tf_x)(y)\leqq -t$. Hence $F(\pm tf_x)=\pm t$ and $y\in Y_t$.

As each Y_s is closed and non-empty $\bigcap_{s>0}Y_s=A_x$ is closed and non-empty.

PROOF OF LEMMA 1.2. Let $y \in A_x$. Without loss of generality we assume that $F(f_x)(y)=1$. By continuity of F we can assume that g is constant in a small neighbourhoodof x, say $g \equiv c$ on $B(x,\delta)$.

Let

$$t > (|c| + \|g\|)\frac{\operatorname{diam}(X)}{\delta}.$$

Then for $x' \notin B(x,\delta)$ we have

$$-t+c \leqq -g(x') \leqq tf_x(x') - g(x') \leqq t - (|c| + \|g\|)\frac{d(x,x')}{\delta} - g(x')$$
$$\leqq t - |c| - \|g\| - g(x') \leqq t - c$$

and for $x' \in B(x,\delta)$ we have $tf_x(x') - c \leqq t - c$ where equality is attained for $x' = x$.

Hence $\|g - tf_x\| = t - c$. Similarly $\|g + tf_x\| = t + c$.

It follows that $|F(g)(y) - t| \leqq t - c$ and $|F(g)(y) + t| \leqq t + c$. Thus $F(g)(y) = c$ and the proof is complete.

PROOF OF LEMMA 1.3. Let $\{y_n\}$ be any sequence in $\bigcup A_x$ with $y_n \in A_{x_n}$ and $\lim y_n = y$. Since X is sequentially compact the sequence $\{x_n\}$ has a convergent subsequence, say $x_n \to x$. By Lemma 1.2 we have $F(tf_x)(y_n) = tf_x(x_n)F(f_{x_n})(y_n)$. If we let $t = 1$ and observe that $f_x(x_n) \to 1$ and $F(t_x)(y_n) \to F(f_x)(y)$ it follows that $\lim F(f_{x_n})(y_n)$ exists and is equal to $F(f_x)(y)$.

Hence $F(tf_x)(y) = \lim F(tf_x)(y_n) = tF(f_x)(y)$ and since $|F(f_x)(y)| = 1$ we have proved that $y \in A_x$.

PROOF OF THEOREM 1.1. Let $K = \bigcup A_x$. By Lemma 1.2 we have for $y \in A_x$

$$F(f+g)(y) = (f(x) + g(x))F(f_x)(y) = F(f)(y) + F(g)(y).$$

Hence the map $f \to F(f)|_K$ is linear. Furthermore we have

$$\|F(f)|_K\| = \sup_{\substack{x \in X \\ A_x \neq \varnothing}} |f(x)| = \sup_{x \in X} |f(x)| = \|f\| \qquad \text{by Lemma 1.1.}$$

Thus we have proved Theorem 1.1.

PROOF OF THEOREM 1.2. Let $K = \bigcup_{x \in X} A_x$ and let $R: C(Y) \to C(K)$ be the restriction map $f \to f|_K$. Since by Lemma 1.1 and Lemma 1.3 K is a closed nonempty subspace of the metric space Y there exist a linear operator $\Lambda: C(K) \to C(Y)$ with $\|\Lambda\| = 1$ and such that $\Lambda(f)|_K = f$ ([5] p, 365, Theorem 21.1.4).

Let $P = \Lambda \circ R$. Clearly P is linear and since $R \circ \Lambda$ is the identity map of $C(K)$ we have $P^2 = \Lambda \circ R \circ \Lambda R = \Lambda \circ R = P$. Hence P is a projection. By Theorem 1.1 $R \circ F$ is linear and since Λ is linear clearly $P \circ F$ is linear. Furthermore, since $\|\Lambda\| = 1$ and by theorem 1.1 we obtain

$$\|P \circ F(f)\| = \|\Lambda(F(f)|_K)\| \leqq \|F(f)|_K\| = \|f\|,$$

$$\|P \circ F(f)\| \geqq \|(P \circ F(f))|_K\| = \|F(f)|_K\| = \|f\|.$$

Thus $P \circ F$: $C(X) \to C(Y)$ is a linear isometry.

REMARK. If we let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ we have $C(X)$ isometric to c and c_0 isometric to $\{f \in C(X);\ f(0) = 0\}$. With some modification of the proofs we see that Theorems 1.1 and 1.2 are also valid if we insert c_0 instead of $C(X)$ and $C(Y)$.

We will now discuss the complex case. If F: $C(X, \mathbf{C}) \to C(Y, \mathbf{C})$ is an isometry then the map $f \to F(f)|_K$ need not be linear for any $K \subset Y$. As a matter of fact, it need not even be real-linear, as the following example shows.

Let $X = [0, 1]$, $Y = [0, 1] \times [0, 2\pi]$ and let P_θ be the orthogonal projection onto the half-plane, $\theta - \pi \leqq \arg z \leqq \theta$, i.e.

$$P_\theta = \begin{cases} z & \text{if } \theta - \pi \leqq \arg z \leqq \theta, \\ e^{i\theta} |z| \cos(\arg z - \theta) & \text{otherwise.} \end{cases}$$

Given $f \in C(X, C)$ we define $F(f)$ on Y by $F(f)(x, \theta) = P_\theta(f(x))$. Clearly F is an isometry.

For θ fixed and $f \equiv e^{i(\theta + \pi/2)}$ we see that $F(f)(x, \theta) = 0$ and $F(-f)(x, \theta) = -e^{i(\theta + \pi/2)}$. Hence the map $f \to F(g)|_{\{(x,\theta)\}}$ is not linear for any point (x, θ) in Y. However, there is a weaker version of Lemma 1.2 for the complex case:

Let $f_{x,\theta} = e^{i\theta} f_x$ and let $A_{x,\theta} = \{y \in Y;\ |F(sf_{x,\theta})(y) - F(-sf_{x,\theta})(y)| = 2s\ \forall s > 0\}$. Then for any $g \in C(X, \mathbf{C})$ with $\arg g(x) = \theta$ we have $F(g)(y) = |g(x)| F(f_{x,\theta})(y)\ \forall y \in A_{x,\theta}$.

In the next theorem we let M be a subset of $C(X)$ and F: $M \to C(Y)$ an isometry with $F(0) = 0$. For $g \in C(X)$ we define $a(g, y)$, $b(g, y)$ on Y by

$$a(g, y) = \sup_{f \in M} \{F(f)(y) - \|f - g\|\}, \qquad b(g, y) = \inf_{f \in M} \{F(f)(y) + \|f - g\|\}.$$

Then we have the following.

THEOREM 1.3. *There exists an isometry* $G: C(X) \to C(Y)$ *with* $G|_M = F$ *if and only if*

(i) *For every* $x \in Y$ *there exists a non-empty set* A_x *such that the* A_x*'s are pairwise disjoint,* $\bigcup_{x \in X} A_x$ *is closed and the map* $\lambda: \bigcup A_x \to X$*, where* $\lambda(y) = x$ *for* $y \in A_x$*, is continuous.*

(ii) *There exists continuous function* $s: \bigcup A_x \to \{-1, 1\}$ *such that*

$$F(f)(y) = s(y)f(x) \qquad \text{for all } y \in A_x \quad \text{and} \quad f \in M.$$

(iii) *For every* $g \in C(X)$ *we have*

$$\overline{\lim_{y \to y_0}}\, a(g, y) \leqq \underline{\lim_{y \to y_0}}\, b(g, y) \qquad \forall y_0 \in Y$$

and

$$\overline{\lim_{y \to y_0}}\, a(g, y) \leqq s(y_0)g(x) \leqq \underline{\lim_{y \to y_0}}\, b(g, y) \qquad \forall y_0 \in A_x.$$

PROOF OF THEOREM 1.3. *The "only if" part.* Let G be an extension of F and let A_x, f_x be as in the beginning of Section 1.

If $y \in A_x$ and $x_1 \neq x$ then by Lemma 1.2.

$$|G(f_{x_1})(y)| = |f_{x_1}(x)G(f_x)(y)| = |f_{x_1}(x)| < 1$$

so $y \notin A_{x_1}$.

To prove that λ is continuous, let $y_0 \in A_{x_0}$ and let $g(x) = d_1(x_0, x)$. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for $y \in B(y_0, \delta) \cap \bigcup A_x$ we have

$$\varepsilon > |G(g)(y) - G(g)(y_0)| = |d_1(x_0, \lambda(y)) \cdot G(f_{\lambda(y)})(y)| = d_1(\lambda(y_0), \lambda(y)).$$

This together with Lemmas 1.1 and 1.3 shows that condition (i) is fulfilled.

Now, let $g \equiv 1$ on X and let $s(y) = G(g)(y)$. By Lemma 1.2 we have for $y \in A_x$, $f \in M$

$$s(y) = G(f_x)(y) = \pm 1 \quad \text{and} \quad F(f)(y) = G(f)(y) = f(x)s(y)$$

and (ii) is proved.

For every $\varepsilon > 0$ we can find $f_1, f_2 \in M$ such that

$$\begin{aligned} a(g, y) &< F(f_1)(y) - \|f_1 - g\| + \varepsilon \\ &\leqq F(f_1)(y) - G(f_1)(y) + G(g)(y) + \varepsilon \\ &= G(g)(y) + \varepsilon \end{aligned}$$

and

$$b(g,y) > F(f_2)(y) + \|f_2 - g\| - \varepsilon \geqq G(g)(y) - \varepsilon.$$

Thus $\overline{\lim}_{y\to y_0} a(g,y) \leqq G(g)(y_0) \leqq \underline{\lim}_{y\to y_0} b(g,y)$ and since $G(g)(y) = s(y)g(x)$ for $y \in A_x$ we see that condition (iii) is satisfied.

The "if" part. We first consider the case when $X \subset Y$, $A_x = \{x\}$, $s \equiv 1$ and prove the following:

(a) Let N be a subset of $C(X)$ containing M. If (iii) is satisfied for all $g \in U$ then there exists an isometrical extension of F to N.

For $g \in N$ we define $A(g,y)$, $B(g,y)$ on Y by

$$A(g,y) = \begin{cases} g(x), & x \in X, \\ \overline{\lim}\, a(g,y), & \text{otherwise}; \end{cases}$$

$$B(g,y) = \begin{cases} g(x), & x \in X, \\ \underline{\lim}\, b(g,y), & \text{otherwise}. \end{cases}$$

Since $\overline{\lim}\, a(g,y)$ is upper semi-continuous (s.c.), X is closed, $g(x)$ is continuous and $\overline{\lim}_{y\to x} a(g,y) \leqq g(x)$ on X, we have that $A(g,y)$ is upper s.c.

Similarly, since $\underline{\lim}\, b(g,y)$ is lower s.c. and $g(x) \leqq \underline{\lim}_{y\to x} b(g,y)$ on X, $B(g,y)$ is lower s.c.

Let h be any fixed function in N and choose $\psi \in C(Y)$ satisfying $A(h,y) \leqq \psi(y) \leqq B(h,y)$ for all $y \in Y$ (this is possible since $A(h,y)$ (upper s.c.) $\leqq B(h,y)$ (lower s.c.), [5]).

Let $M_1 = M \cup \{h\}$, $F(h) = \psi$ and define

$$a_1(g,y) = \sup_{f\in M_1} \{F(f)(y) - \|f-g\|\}, \qquad b_1(g,y) = \inf_{f\in M_1} \{F(f)(y) + \|f-g\|\}.$$

We will now prove the following:

(b) The map $F\colon M_1 \to C(Y)$ is an isometry and for $A_x = \{x\}$, $s \equiv 1$ condition (ii), (iii) is satisfied for all $g \in N$.

Once (b) is proved (a) follows by iterating the procedure with a dense sequence $\{h_n\}$ in N.

In order to prove (b) we first show that $a(g,y_0) \leqq \overline{\lim}_{y\to y_0} a(g,y)$ and $\underline{\lim}_{y\to y_0} b(g,y) \leqq b(g,y_0)$ for all $y \in Y$ and $g \in C(X)$.

Given $\varepsilon > 0$ we can find $f \in M$ such that $a(g,y_0) < F(f)(y_0) - \|f-g\| + \varepsilon/2$

and $\delta > 0$ such that $|F(f)(y) - F(f)(y_0)| < \varepsilon/2$ for all $y \in B(y_0, \delta)$. It follows that

$$\begin{aligned} a(g, y_0) &< F(f)(y) - \|f - g\| + F(f)(y_0) - F(f)(y) \\ &< F(f)(y) - \|f - g\| + \varepsilon \\ &\leqq a(g, y) + \varepsilon \qquad \text{on } B(y_0, \delta). \end{aligned}$$

Thus $a(g, y_0) \leqq \overline{\lim}_{y \to y_0} a(g, y)$. Similarly we get $\underline{\lim}_{y \to y_0} b(g, y) \leqq b(g, y_0)$.

Let $f \in M$. For $y_0 \in Y \setminus X$ we have

$$F(f)(y_0) - \psi(y_0) \leqq F(f)(y_0) - \overline{\lim_{y \to y_0}}\, a(h, y) \leqq F(f)(y_0) - a(h, y_0) \leqq \|f - h\|$$

and

$$\psi(y_0) - F(f)(y_0) \leqq \underline{\lim}_{y \to y_0}\, b(h, y) - F(f)(y_0) \leqq b(h, y_0) - F(f)(y_0) \leqq \|f - h\|.$$

For $x \in X$ we have by definition $\psi(x) = h(x)$ and by condition (ii) $F(f)(x) = f(x)$. Hence $\|F(f) - \psi\| = \|f - h\|$ for all $f \in M$.

It remains to prove that condition (iii) is satisfied for all $g \in N$. Clearly

$$\overline{\lim}\, a_1(g, y) = \sup\{\overline{\lim}\, a(g, y), \psi(y) - \|h - g\|\}$$

and

$$\underline{\lim}\, b_1(g, y) = \inf\{\underline{\lim}\, b(g, y), \psi(y) + \|h - g\|\}.$$

Now, for $g \in N$ and $x \in X$ we have

$$\overline{\lim_{y \to x}}\, a(g, y) \leqq g(x) \leqq \underline{\lim}_{y \to x}\, b(g, y).$$

Thus we have

$$\overline{\lim_{y \to x}}\, a_1(a, y) \leqq \sup\{g(x), h(x) - \|h - g\|\} = g(x)$$

and

$$\underline{\lim}_{y \to x}\, b_1(g, y) \geqq \inf\{g(x), h(x) + \|h - g\|\} = g(x).$$

For $y \in Y \setminus X$ we need to prove

$$\overline{\lim}\, a_1(g, y) \leqq \psi(y) + \|h - g\| \quad \text{and} \quad \psi(y) - \|h - g\| \leqq \underline{\lim}\, b_1(g, y).$$

Given $\varepsilon > 0$ we can find $f \in M$ such that

$$\begin{aligned} a(g,y) - a(h,y) &\leqq F(f)(y) - \|f-g\| + \varepsilon - a(h,y) \\ &\leqq F(f)(y) - \|f-g\| + \varepsilon - F(f)(y) + \|f-h\| \\ &\leqq \|g-h\| + \varepsilon. \end{aligned}$$

Hence

$$\overline{\lim}(a(g,y) - a(h,y)) \leqq \|h-g\|.$$

Similarly

$$\overline{\lim}(b(h,y) - b(g,y)) \leqq \|h-g\|.$$

Therefore we have

$$\overline{\lim}\, a(g,y) - \psi(y) \leqq \overline{\lim}\, a(g,y) - \overline{\lim}\, a(h,y) \leqq \overline{\lim}(a(g,y) - a(h,y)) \leqq \|h-g\|$$

and

$$\psi(y) - \underline{\lim}\, b(g,y) \leqq \underline{\lim}\, b(h,y) - \underline{\lim}\, b(g,y) \leqq \overline{\lim}(b(h,y) - b(g,y)) \leqq \|h-g\|.$$

Thus condition (iii) is satisfied.

This proves (b) and so the proof of (a) is complete.

We will now use (a) to prove the general thoėrem. Let $\bigcup A_x = Z \subset Y$. By condition (i) we can define a linear isometry $L\colon C(X) \to C(Z)$ by $L(f)(z) = s(z)f(x)$ for $z \in A_x$.

Now let $F_1\colon L(M) \to C(Y)$ be defined by $F_1(L(f)) = F(f)$. Clearly F_1 is an isometry satisfying condition (i), (ii) with $A_z = \{z\}$ and $s \equiv 1$. Moreover, condition (iii) is satisfied for all $h \in L(C(X))$.

To see this, let $h = L(g)$. Then we have

$$\sup_{L(f) \in L(M)} \{F_1(L(f))(y) - \|L(f) - L(g)\|\} = a(g,y)$$

and

$$\inf_{L(f) \in L(M)} \{F_1(L(f))(y) - \|L(f) - L(g)\|\} = b(g,y).$$

For $z \in A_x$ and $g \in C(X)$ we have

$$\overline{\lim_{y \to z}}\, a(g,y) \leqq s(z)g(x) = L(g)(z) \leqq \underline{\lim}_{y \to z}\, b(g,y).$$

Hence condition (iii) is satisfied for all $h \in L(C(X))$ and with $s \equiv 1$. Thus we can find an isometrical extension $\tilde{F}_1$ of F_1 to $L(C(X))$. Clearly $\tilde{F}_1 \circ L$ is an isometrical extension of F to $C(X)$ and the proof of Theorem 1.3 is complete.

REMARK 1. One can easily see that the general case can be carried out directly in almost the same way as we carried out (a) by defining

$$A(g, y) = \begin{cases} s(y)g(x), & y \in A_x \\ \overline{\lim}\, a(g, y), & \text{otherwise} \end{cases}$$

and similarly for $B(g, y)$. However the structure of the general isometry may appear in a more clear way when carrying out (a) first.

REMARK 2. If we let c_0 replace $C(X)$ and $C(Y)$ in Theorem 1.3 and exclude the requirement of s to be continuous we have an extension theorem for isometries from subsets of c_0. In the proof we then let $A_0 = \{0\}$ and $s(0) = 1$. In this case we have

$$\overline{\lim_{y \to y_0}}\, a(g, y) = a(g, y_0) \leqq b(g, y_0) = \underline{\lim_{y \to y_0}}\, b(g, y) \qquad \text{for all } y_0 \neq 0$$

and since $a(g, y) \leqq s(y)g(y_0) \leqq b(g, y_0)$ for $y_0 \in A_x$ the condition (iii) reduces to:

$$\text{For all } g \in C(X) \qquad \text{we have } \overline{\lim_{y \to 0}}\, a(g, y) \leqq 0 \leqq \underline{\lim_{y \to 0}}\, b(g, y).$$

2. Almost isometries from the unit ball of $C(X)$ into the unit ball of $C(Y)$

Let X, Y be compact metric spaces with metrics d_1 and d_2 and let $B_R(C(X))$ denote the ball in $C(K)$ with center 0 and radius R.

THEOREM 2.1. *Let F: $B_1(C(X)) \to B_1(C(Y))$ with $F(0) = 0$ and*

$$(1 - \varepsilon)\|f - g\| \leqq \|F(f) - F(g)\| \leqq (1 + \varepsilon)\|f - g\| \qquad \textit{for all } f, g \in B_1(C(X)).$$

Then there exists an isometry G: $B_{1-\delta_1(\varepsilon)}(C(X)) \to B_1(C(Y))$ such that

$$\|G(f) - F(f)\| < \delta_2(\varepsilon) \qquad \textit{on } B_{1-\delta_1(\varepsilon)}(C(X))$$

where $\delta_1(\varepsilon) \to 0$ and $\delta_2(\varepsilon) \to 0$ when $\varepsilon \to 0$.

The proof is based on the following Proposition. Let a be fixed, $16\varepsilon/(1 + \varepsilon) < a \leqq 1$.

PROPOSITION. *There is a closed set $K \subset Y$, a continuous onto map ϕ: $K \to X$ and a continuous sign function s: $K \to \{-1,1\}$ with the following property:*

If $f_1, f_2 \in B_{1-a/2}(C(X))$ and $\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$ then there is $y_0 \in \phi^{-1}(x_0)$ so that

$$|F(f_i)(y_0) - (1+\varepsilon)s(y_0)f_i(x_0)| \leqq 3\varepsilon, \qquad i = 1,2.$$

DEFINITION 1. Given $x_0 \in X$ we say that $f \in C(X)$ is a tentfunction at x_0 if for some $\delta > 0$

$$f(x) = \begin{cases} 1 - \dfrac{d_1(x_0, x)}{\delta}, & x \in B(x_0, \delta), \\ 0, & \text{otherwise}. \end{cases}$$

For the proof of the Proposition we need some lemmas.

LEMMA 2.1. *Let $\{f_n\} \subset C(X)$, $\{x_n\} \subset X$, $\{y_n\} \subset Y$ be sequences with $y_n \to y$ and f_n a tentfunction at x_n with* $\text{supp}(f_n) = B(x_n, \delta_n)$ *where $\delta_n \to 0$ when $n \to \infty$. If for all n*

$$\text{(I)} \qquad 2a(1+\varepsilon) - 4\varepsilon \leqq |F(af_n)(y_n) - F(-af_n)(y_n)|$$

then $\lim_{n\to\infty} x_n$ *exists.*

DEFINITION 2. We say that $y \in A_x$ if there exist sequences $\{f_n\}$, $\{x_n\}$, $\{y_n\}$ satisfying the conditions in Lemma 2.1 with $x = \lim x_n$ and $y = \lim y_n$.

PROOF OF LEMMA 2.1. Clearly $\{x_n\}$ contains a convergent subsequence, say $\{x_{n'}\}$, with $\lim x_{n'} = x$. Assume that $\{x_n\}$ is not convergent. Then for some $d > 0$ there exist, for every N, $n \geqq N$ such that $d_1(x_n, x) \geqq d$. Let $g \in C(X)$ with $0 \leqq g \leqq a/2$, $g \equiv a/2$ on $B(x, d/4)$ and with $\text{supp}(g) \subset B(x, d/2)$.

Now, for every N it is possible to find $n, n' \geqq N$ such that $\text{supp}(f_{n'}) \subset B(x, d/4)$ and $\text{supp}(f_n) \cap B(x, d/2) = \emptyset$. Then we have $\|g - af_{n'}\| = a/2$, $\|g + af_{n'}\| = 3a/2$ and $\|g \pm af_n\| = a$.

Therefore we get

$$F(af_{n'})(y_{n'}) - \frac{a}{2}(1+\varepsilon) \leqq F(g)(y_{n'}) \leqq F(af_{n'})(y_{n'}) + \frac{a}{2}(1+\varepsilon),$$

$$F(-af_{n'})(y_{n'}) - \frac{3a}{2}(1+\varepsilon) \leqq F(g)(y_{n'}) \leqq F(-af_{n'})(y_{n'}) + \frac{3a}{2}(1+\varepsilon),$$

$$F(\pm af_n)(y_n) - a(1+\varepsilon) \leqq F(g)(y_n) \leqq F(\pm af_n)(y_n) + a(1+\varepsilon).$$

By hypothesis of F and by (I) we have for all k

$$a(1+\varepsilon)-4\varepsilon \leqq F(af_k)(y_k) \leqq a(1+\varepsilon),$$

$$-a(1+\varepsilon) \leqq F(-af_k)(y_k) \leqq -a(1+\varepsilon)+4\varepsilon,$$

or

$$-a(1+\varepsilon) \leqq F(af_k)(y_k) \leqq -a(1+\varepsilon)+4\varepsilon,$$

$$a(1+\varepsilon)-4\varepsilon \leqq F(-af_k)(y_k) \leqq a(1+\varepsilon).$$

We obtain that

$$-4\varepsilon \underset{(-)}{+} \frac{a}{2}(1+\varepsilon) \leqq F(g)(y_{n'}) \leqq \underset{(-)}{+} \frac{a}{2}(1+\varepsilon)+4\varepsilon$$

and

$$-4\varepsilon \leqq F(g)(y_n) \leqq 4\varepsilon.$$

Thus we have

$$|F(g)(y_{n'})| \geqq \frac{a}{2}(1+\varepsilon)-4\varepsilon \quad \text{and} \quad |F(g)(y_n)| \leqq 4\varepsilon.$$

Since $F(g) \in C(Y)$, $16\varepsilon/(1+\varepsilon) < a$ fixed and $d_2(y_{n'}, y_n) \to 0$ when $n, n' \to \infty$ this clearly gives a contradiction for n, n' large enough. Hence $\{x_n\}$ is convergent.

As a consequence of Lemma 2.1 we have the following.

LEMMA 2.2. *The sets A_x are pairwise disjoint.*

PROOF. Let $y \in A_{x_1} \cap A_{x_2}$ and let $\{f_{in}\}$, $\{x_{in}\}$ and $\{y_{in}\}$ be corresponding sequences in the definition of A_{x_i}, $i = 1, 2$. Since $\lim y_{1n} = \lim y_{2n}$ the sequences $\{f_{11}, f_{21}, f_{12}, f_{22}, \ldots\}$, $\{x_{11}, x_{21}, x_{12}, \ldots\}$ and $\{y_{11}, y_{21}, y_{12}, \ldots\}$ clearly satisfy the conditions in Lemma 2.1. Hence $\{x_{11}, x_{21}, x_{12}, \ldots\}$ is convergent and $x_1 = \lim x_{1n} = \lim x_{2n} = x_2$.

LEMMA 2.3. *If $y_k \in A_{x_k}$ and $y_k \to y$ then $x_k \to x$ and $y \in A_x$.*

PROOF. Let $\{f_{kn}\}$, $\{x_{kn}\}$ and $\{y_{kn}\}$ correspond to y_k in the definition of A_{x_k}. For every k we can find $n(k)$ such that $d_1(x_{kn(k)}, x_k) < 1/k$, $d_2(y_{kn(k)}, y_n) < 1/k$ and $\operatorname{supp}(f_{kn(k)}) \subset B(x_k, 1/k)$.

Then the sequences $\{f_{kn(k)}\}$, $\{x_{kn(k)}\}$ and $\{y_{kn(k)}\}$ clearly satisfy the conditions in Lemma 2.1.

Thus $\exists \lim_{k\to\infty} x_{kn(k)} = x$ and $\lim_{k\to\infty} y_{kn(k)} = y \in A_x$.

LEMMA 2.4. *Let $y \in A_x$ and let $\{f_n\}$, $\{x_n\}$ and $\{y_n\}$ be any collection of sequences satisfying the conditions in Lemma 2.1. Then*

$$\lim_{n\to\infty} \operatorname{sign} F(af_n)(y_n) = \operatorname{sign} F\left(\frac{a}{2}\right)(y).$$

PROOF. For each y_n we have $\operatorname{sign}(af_n)(y_n) = \operatorname{sign}(a/2)(y_n)$ and $|F(a/2)(y_n)| > 4\varepsilon$. Indeed, by definition we have

$$|F(af_n)(y_n)| \geqq 2(1+\varepsilon)a - 4\varepsilon - |F(-af_n)(y_n)| \geqq (1+\varepsilon)a - 4\varepsilon$$

and

$$F(af_n)(y_n) - (1+\varepsilon)\frac{a}{2} \leqq F\left(\frac{a}{2}\right)(y_n) \leqq F(af_n)(y_n) + (1+\varepsilon)\frac{a}{2}.$$

Hence

$$F\left(\frac{a}{2}\right)(y_n) \geqq (1+\varepsilon)\frac{a}{2} - 4\varepsilon > 4\varepsilon \qquad \text{if } F(af_n)(y_n) > 0.$$

Similarly,

$$F\left(\frac{a}{2}\right)(y_n) < -4\varepsilon \qquad \text{if } F(af_n)(y_n) < 0.$$

Thus $\operatorname{sign} F(a/2)(y)$ is well-defined and equal to $\lim_{n\to\infty} \operatorname{sign} F(af_n)(y_n)$.

PROOF OF PROPOSITION. Let $K = \bigcup A_x$, $s(y) = \operatorname{sign} F(a/2)(y)$ on K and let $\phi\colon K \to X$ be defined by $\phi(y) = x$ for $y \in A_x$. By Lemmas 2.2 and 2.3 we see that K is closed and s, ϕ is well-defined and continuous.

Let $f_1, f_2 \in B_{1-a/2}(C(X))$ and let $\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$. In order to prove that there is a $y_0 \in K$ with $\phi(y_0) = x_0$ and

$$|F(f_i)(y_0) - (1+\varepsilon)s(y_0)f_i(x_0)| \leqq 3\varepsilon$$

we construct two sequences of functions in the following way.

Let d be such that $|f_i(x) - f_i(x_0)| \leqq a$ on $B(x_0, d)$. For $n = 1, 2, 3, \ldots$ we define

$$p_n(x) = \begin{cases} 1 - \dfrac{nad_1(x, x_0)}{d} & \text{on } B\left(x_0, \dfrac{d}{n}\right) \\[2ex] \min\limits_{i=1,2} \{1 - f_i(x_0) + f_i(x), 1 - a\} & \text{otherwise} \end{cases}$$

and

$$q_n(x) = \begin{cases} -1+\dfrac{nad_1(x,x_0)}{d} & \text{on } B\left(x_0, \dfrac{d}{n}\right) \\ \max\limits_{i=1,2}\{-1-f_i(x_0)+f_i(x), -1+a\} & \text{otherwise.} \end{cases}$$

Clearly, $p_n, q_n \in B_1(C(X))$ and we have

$$(*) \qquad \|f_i - p_n\| \to 1 - f_i(x_0) \quad \text{and} \quad \|f_i - q_n\| \to 1 + f_i(x_0) \qquad \text{when } n \to \infty.$$

By continuity this is obvious on $B(x_0, d/n)$. For $x \notin B(x_0, d/n)$ we have $p_n(x) - f_1(x) \leqq 1 - f_1(x_0)$ and if $p_n(x) = 1 - f_1(x_0) + f_1(x)$ then

$$f_1(x) - p_n(x) = -1 + f_1(x_0) \leqq f(x_0) - 1.$$

If $p_n(x) = 1 - a$ then

$$f_1(x) - p_n(x) \leqq 1 - a/2 - (1-a) = 1 - \left(1 - \frac{a}{2}\right) \leqq 1 - f_1(x_0).$$

Finally, if $p_n(x) = 1 - f_2(x_0) + f_2(x)$ then $f_2(x) - f_1(x) \leqq f_2(x_0) - f_1(x_0)$. Since

$$|f_1(x) - f_2(x)| \leqq |f_1(x_0) - f_2(x_0)|$$

this implies

$$f_2(x) - f_1(x) = f_2(x_0) - f_1(x_0) < 0 \quad \text{or} \quad f_1(x) - f_2(x) \leqq f_2(x_0) - f_1(x_0).$$

Hence $f_1(x) - p_n(x) = f_1(x_0) - 1 \leqq 1 - f_1(x_0)$ or

$$f_1(x) - p_n(x) = f_1(x) - f_2(x) - 1 + f_2(x_0) \leqq -1 + 2f_2(x_0) - f_1(x_0) \leqq 1 - f_1(x_0).$$

Thus we have $|p_n(x) - f_1(x)| \leqq 1 - f(x_0)$ on $X \setminus B(x_0, d/n)$.

Similarly, $|q_n(x) - f_1(x)| \leqq 1 + f(x_0)$ on $X \setminus B(x_0, d/n)$.

Since $\|p_n - q_n\| = 2$ there exist y_n for every n such that

$$(**) \qquad 2(1-\varepsilon) \leqq |F(p_n)(y_n) - F(q_n)(y_n)| \leqq 2(1+\varepsilon).$$

The sequence $\{y_n\}$ contains a convergent subsequence, say $y_n \to y_0$. We shall now prove that $y_0 \in A_{x_0} = \phi^{-1}(x_0)$.

Consider the functions r_n defined by

$$r_n(x) = \begin{cases} 1 - \dfrac{nd_1(x,x_0)}{d} & \text{on } B\left(x_0, \dfrac{d}{n}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Since r_n is a tentfunction at x_0, $d/n \to 0$ and $y_n \to y_0$ we have $y_0 \in \phi^{-1}(x_0)$ if we can prove that $2a(1+\varepsilon) - |F(ar_n)(y_n) - F(-ar_n)(y_n)| \leqq 4\varepsilon$.

Assume that $F(p_n)(y_n) \geqq F(q_n)(y_n)$. From (**) we obtain $2(1-\varepsilon) \leqq F(p_n)(y_n) - F(q_n)(y_n)$ and since $\|p_n - ar_n\| = \|q_n + ar_n\| = 1-a$ we get

$$
\begin{aligned}
-|F(ar_n)(y_n) - F(-ar_n)(y_n)| &\leqq F(-ar_n)(y_n) - F(ar_n)(y_n) \\
&\leqq F(-ar_n)(y_n) - F(q_n)(y_n) + F(p_n)(y_n) \\
&\quad - F(a_n r_n)(y_n) + F(q_n)(y_n) - F(p_n)(y_n) \\
&\leqq (1+\varepsilon)(1-a) + (1+\varepsilon)(1-a) - 2(1+\varepsilon) \\
&= -2a(1+\varepsilon) + 4\varepsilon.
\end{aligned}
$$

Thus $y_0 \in \phi^{-1}(x_0)$.

The case $F(p_n)(y_n) \leqq F(q_n)(y_n)$ is proved similarly. Since $|F(p_n)(y_n)| \leqq 1$ and $|F(q_n)(y_n)| \leqq 1$, (**) implies

$$
\begin{aligned}
1 - 2\varepsilon &\leqq F(p_n)(y_n) \leqq 1, \\
-1 &\leqq F(q_n)(y_n) \leqq -1 + 2\varepsilon,
\end{aligned}
$$

or

$$
\begin{aligned}
-1 &\leqq F(p_n)(y_n) \leqq -1 + 2\varepsilon, \\
1 - 2\varepsilon &\leqq F(q_n)(y_n) \leqq 1.
\end{aligned}
$$

One can easily check that $\operatorname{sign} F(p_n)(y_n) = \operatorname{sign} F(ar_n)(y_n)$ so for n large enough we have $\operatorname{sign} F(p_n)(y_n) = s(y_0)$. Hence for n large enough those inequalities can be rewritten in the form

$$
\begin{aligned}
1 - 2\varepsilon &\leqq s(y_0) F(p_n)(y_n) \leqq 1, \\
-1 &\leqq s(y_0) F(q_n)(y_n) \leqq -1 + 2\varepsilon.
\end{aligned}
$$

From (*) we obtain

$$
\begin{aligned}
-\varepsilon(n, f_i) + F(p_n)(y_n) - (1+\varepsilon)(1 - f_i(x_0)) &\leqq F(f_i)(y_n) \\
&\leqq (1+\varepsilon)(1 - f_i(x_0)) + F(p_n)(y_n) + \varepsilon(n, f_i), \\
-\varepsilon(n, f_i) + F(q_n)(y_n) - (1+\varepsilon)(1 + f_i(x_0)) &\leqq F(f_i)(y_n) \\
&\leqq (1+\varepsilon)(1 + f_i(x_0)) + F(q_n)(y_n) + \varepsilon(n, f_i),
\end{aligned}
$$

where $\varepsilon(n, f_i) \to 0$ when $n \to \infty$.

Hence for n large enough we have

$$-\varepsilon(n,f_i)-3\varepsilon+(1+\varepsilon)s(y_0)f_i(x_0)\leqq F(f_i)(y_n)\leqq(1+\varepsilon)s(y_0)f_i(x_0)+3\varepsilon+\varepsilon(n,f_i).$$

Letting $n\to\infty$ we obtain $|F(f_i)(y_0)-(1+\varepsilon)s(y_0)f_i(x_0)|\leqq 3\varepsilon$ and the proof is complete.

Proof of Theorem 2.1. Let ϕ and s be as in the Proposition. Since $s: K\to\{-1,1\}$ and K is closed we can find, by Urysohn's lemma, a continuous function $\bar{s}: Y\to[-1,1]$ with $\bar{s}\,|_K = s$.

Now, let $M_1(X)$ be the unit ball of Radon measures on X endowed with the weak*-topology. Define a set valued map on Y by

$$\psi(y)=\{s(y)\delta_{\phi(y)}\} \qquad \text{if } y\in K$$

and

$$\psi(y)=\{\bar{s}(y)\mu;\ \mu \text{ probability measure} \in M_1(X)\} \qquad \text{if } y\in Y\setminus K.$$

Clearly $\psi(y)$ is a closed and convex subset of $M_1(X)$ for all $y\in X$. Furthermore, one can easily check that the set $\{y\in Y; \psi(y)\cap G\neq\emptyset\}$ is open in Y for every open set G in $M_1(X)$. Hence, using Michael's selection theorem ([4] p. 169) we can find a continuous map φ: $Y\to M_1(X)$ such that $\varphi(y)=s(y)\delta_{\phi(y)}$ on K.

Now given any $y\in Y$ and $f\in B_{1-\alpha/2}(C(K))$ we define

$$G(f)(y)=\sup\left\{\inf\left\{\varphi(y)(f),\frac{1}{1+\varepsilon}(F(f)(y)+3\varepsilon)\right\},\frac{1}{1+\varepsilon}(F(f)(y)-3\varepsilon)\right\}.$$

We observe that $|F(f)(y)-(1+\varepsilon)\varphi(y)(f)|\leqq 3\varepsilon$ if and only if $G(f)(y)=\varphi(y)(f)$.

Since φ is weak*-continuous we have $\varphi(y)(f)$ continuous on Y and hence $G(f)\in C(Y)$.

Furthermore we have

$$\begin{aligned}|F(f)(y)-G(f)(y)|&=|F(f)(y)-\varphi(y)(f)|\\&\leqq|F(f)(y)-(1+\varepsilon)\varphi(y)(f)|+\varepsilon\,|\varphi(y)(f)|\\&\leqq 4\varepsilon\end{aligned}$$

or

$$\begin{aligned}|F(f)(y)-G(f)(y)|&=\left|F(g)(y)-\frac{1}{1+\varepsilon}(F(f)(y)\pm3\varepsilon)\right|\\&\leqq\frac{1}{1+\varepsilon}(|\varepsilon F(f)(y)|+3\varepsilon)\\&\leqq 4\varepsilon.\end{aligned}$$

Thus $\|F(f)-G(f)\| \leqq 4\varepsilon$ on $B_{1-a/2}(C(X))$.

We now prove that G is an isometry and to do this we first show that

$$|G(f_1)(y)-G(f_2)(y)| \leqq \|f_1-f_2\| \qquad \forall y \in Y.$$

This clearly holds if $G(f_i)(y)=\varphi(y)(f_i)$, $i=1,2$ or

$$G(f_i)(y)=\frac{1}{1+\varepsilon}\left(F(f_i)(y) \underset{(+)}{-} 3\varepsilon\right), \qquad i=1,2.$$

If

$$G(f_1)(y)=\frac{1}{1+\varepsilon}(F(f_1)(y)-3\varepsilon) \quad \text{and} \quad G(f_2)(y)=\frac{1}{1+\varepsilon}(F(f_2)(y)+3\varepsilon)$$

then by definition

$$G(f_1)(y) \geqq \varphi(y)(f_1) \quad \text{and} \quad G(f_2)(y) \leqq \varphi(y)(f_2).$$

Hence $G(f_2)(y)-G(f_1)(y) \leqq \varphi(y)(f_2-f_1) \leqq \|f_1-f_2\|$ and

$$G(f_1(y)-G(f_2)(y)=\frac{1}{1+\varepsilon}(F(f_1)(y)-F(f_2)(y)-6\varepsilon) \leqq \|f_1-f_2\|.$$

If

$$G(f_1)(y)=\varphi(y)(f_1) \quad \text{and} \quad G(f_2)(y)=\frac{1}{1+\varepsilon}(F(f_2)(y)-3\varepsilon)$$

then

$$G(f_1)(y)-G(g_2)(y) \leqq \varphi(y)(f_1)-\varphi(y)(f_2).$$

Furthermore, we have

$$(1+\varepsilon)\varphi(y)(f_1)-3\varepsilon \leqq F(f_1)(y) \leqq (1+\varepsilon)\varphi(y)(f_1)+3\varepsilon$$

and

$$F(f_1)(y)-(1+\varepsilon)\|f_1-f_2\| \leqq F(f_2)(y) \leqq F(f_1)(y)+(1+\varepsilon)\|f_1-f_2\|.$$

Using the right side of those inequalities we get

$$\frac{1}{1+\varepsilon}(F(f_2)(y)-3\varepsilon)-\varphi(y)(f_1) \leqq \|f_1-f_2\|.$$

Finally, if we use the left side we obtain a proof for the remaining case,

$$G(f_1)(y)=\varphi(y)(f_1) \quad \text{and} \quad G(f_2)(y)=\frac{1}{1+\varepsilon}(F(f_2)(y)+3\varepsilon).$$

Hence $\| G(f_1) - G(f_2) \| \leqq \| f_1 - f_2 \|$.

Now if $\| f_1 - f_2 \| = | f_1(x_0) - f_2(x_0) |$, then by the Proposition we can find a point $y_0 \in \phi^{-1}(x_0)$ such that

$$3\varepsilon \geqq | F(f_i)(y_0) - (1+\varepsilon) s(y_0) f_i(x_0) | = | F(f_i)(y_0) - (1+\varepsilon)\varphi(y_0)(f_i) |.$$

Thus $G(f_i)(y_0) = \varphi(y_0)(f_i) = s(y_0) f_i(x_0)$ so we have

$$\| G(f_1) - G(f_2) \| \geqq | s(y_0)(f_1(x_0) - f_2(x_0)) | = \| f_1 - f_2 \|.$$

Since we may choose $a = 16\varepsilon$, $\delta_1(\varepsilon) = a/2$ and $\delta_2(\varepsilon) = 4\varepsilon$ the proof of Theorem 2.1 is complete.

REMARK. With some modifications of the proof we see that Theorem 2.1 is also valid if we let c_0 replace $C(X)$ and $C(Y)$.

However, from the proof we cannot draw any conclusion whether Theorem 2.1 is valid or not in the complex case. For example, we have nothing corresponding to the auxiliary functions p_n, q_n in this case.

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